2.2 - Factorial Notation & Permutations

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Consider how many ways you can organize 5 objects side by side. You have 5 options for the first stage, 4 for the next, etc.

Using the fundamental counting principle, we would quickly determine the solution to be the product:

A B D E

5 4 3 2 1

a : 5
b : 4
c : 3
d : 2
e : 1

$5 \times 4 \times 3 \times 2 \times 1 = 120$ possibilities

In math, there's a notation we can use to describe products like this. It's called "**factorial notation**" and it's written as:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$

This is read as "five factorial," or "factorial five."

Example

Calculate the following without using a calculator.

i) $4! = 4 \times 3 \times 2 \times 1$

$= 24$

ii) $3! \times 2!$

$3 \times 2 \times 1 \times 2 \times 1 = 12$

Note that we can rewrite factorials a number of different ways if they are large enough:

6! = $6 \times 5 \times 4 \times 3 \times 2 \times 1$

6! = $6 \times 5!$

6! = $6 \times 5 \times 4!$

6! = $6 \times 5 \times 4 \times 3!$

etc....

Using this strategy can be useful to simplify quotients.

$$\frac{6!}{4!} = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}} = 6 \times 5 = 30$$

You can also write certain products as a quotient of factorials:

$$6 \times 5 \times 4 = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!}} =$$

Example

Simplify $\frac{8!}{5!3!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{3!\,\cancel{5!}} = \frac{8 \times 7 \times 6}{3!} = \frac{8 \times 7 \times \cancel{6}}{\cancel{6}}$

$$= 8 \times 7 = \boxed{56}$$

Generally, factorials can be defined as:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times (3) \times (2) \times (1), \text{ where } n \in \mathbb{N}$$

($n \in \mathbb{N}$ means "n is an element of the natural numbers," and natural numbers are whole numbers from 1 to infinity)

Note: 0! is defined to be equal to 1.

Example

next lowest NEXT lowest

Simplify $\frac{(n+2)!}{n!} = \frac{(n+2)\times(n+1)\times\cancel{n!}}{\cancel{n!}} = (n+2)(n+1)$

Simplify $\frac{(n-2)!}{(n-1)!} = \frac{\cancel{(n-2)!}}{(n-1)\times\cancel{(n-2)!}} = \frac{1}{n-1}$

Simplify $\frac{n!}{n(n-1)} = \frac{n\times\cancel{(n-1)}\times(n-2)!}{n\cancel{(n-1)}} = (n-2)!$

Example

Solve $\frac{n!}{(n-2)!} = 42$

$\frac{n\times(n-1)\times\cancel{(n-2)!}}{\cancel{(n-2)!}} = 42$

$n(n-1) = 42$

You can solve as a quadratic, or determine what 2 consecutive numbers = 42. The larger one is "n".

↳ $6\times7=42$.

$\boxed{\therefore n=7}$

Let's say your bank PIN is 3-digits long, and you've forgotten it, except for the fact you know it contains the numbers 3, 5, and 7.

List the number of possible PINs you can try to access your account (there's six):

357 537 735
375 573 753

As you know, the **order** of the digits of your PIN matter. That is, if the order is incorrect, you can't access your account. An arrangement like this where the order of the elements is important is called a permutation.

Permutations of "n" Different Elements Taken "n" at a Time

List all the arrangements of the letters of the word "cat"

cat act tac
cta atc tca = 6

Write this number in factorial notation.

3!

We already know this pattern, but to put it in words:

"The number of permutations of *"n"* different elements taken all at the same time is equal to n!"

Contrast the last problem with this one:

Determine the number of three letter arrangements (not necessarily complete words) you can make from the letters in "graphite."

Clearly we can't use the strategy from above, we're not taking all the elements at the same time.

In this example, the total number of elements is n = 8

The total number of elements we're taking at a time is r = 3

We use a special notation to describe situations like this: ${}_nP_r$

So in this example, we would write it as: ${}_8P_3$

The formula for solving this is:

"The number of Permutations of "n" Different Elements Taken "r" at a Time is:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

So, to solve the problem we can set up the following:

$$ {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}} = 336 $$

336 possible 3 letter words.

Example

How many numbers (up to a max of 4 digits) can be made from the digits 2, 3, 4, and 5 if no digit can be repeated? (Hint: Consider 1, 2, 3, and 4 digit numbers separately)