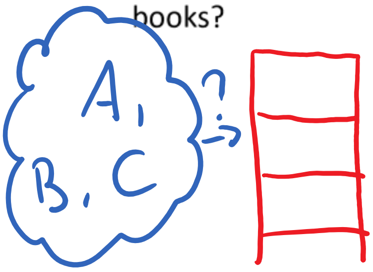
2.4 - Permutations with Repetitions

August 29, 2019 1:41 PM

Let's say a book store is creating a display with a 3-shelf stand where one book can be placed on each shelf: Book A, book B, and book C. How many ways can we arrange these books?



$_3P_3 = 3! = 6$

The store sells out of book C, and replaces the display copy with another book A. Sketch the possible arrangements of the new display. How many are there?

A, A, B
A, B, A
B, A, A

3 possibilities, not 6.

ie. we can't simply replace all C's with A's because we get identical combinations.

In this case, the solution to the second problem is the solution to the first problem divided by $\underline{\ 2 = 2!\ }$.

Let's consider something from the previous chapters. How many arrangements of the letters from the word "ROSE" can we make?

We know now that this would be $_4P_4 = 24$ permutations:

ROSE	REOS	OSRE	SROE	SERO	EORS
ROES	RESO	OSER	SREO	SEOR	EOSR
RSOE	ORSE	OERS	SORE	EROS	ESRO
RSEO	ORES	OESR	SOER	ERSO	ESOR

If you haven't noticed by now, all these words you've seen in class and from your homework never contained a repeated letter

Let's see what happens when we have a word that does.

Let's change the "E" from "ROSE" to another "S" to create the word "ROSS." We can change every "E" from the previous chart to an "S" to get every possible permutation of the word "ROSS."

etc....

ROSS	RSOS	OSRS	SROS	SSRO	SORS
ROSS	RSSO	OSSR	SRSO	SSOR	SOSR
RSOS	ORSS	OSRS	SORS	SROS	SSRO
RSSO	ORSS	OSSR	SOSR	SRSO	SSOR

Highlight some of the identical "words".

As we can see, previous permutations such as "ROSE" and "ROES" both become "ROSS" in the new problem. Since it is the same word, we only count it as one permutation.

Clearly there are less possible permutations now that we have a repeated letter. If we count them up, we'll find there are 12 permutations of "ROSS."

That's $\frac{1}{2}$ or $\frac{1}{2!}$ as many permutations are there was for "ROSE."

Therefore, we can say that the total number of permutations for "ROSS" is $\frac{4!}{2!}$.
We do this to make sure we don't double count the identical words.

This pattern has a nice formula to help us out:

When dealing with permutations with repetitions,

$$Number\ of\ permutations = \frac{n!}{a! \times b! \times c! \times \ldots}$$

where n = total number of objects
a = number of repeated objects
b = number of repeated different objects
c = number of repeated further different objects
etc.

Example

Change "ROSS" to "RSSS." How many different permutations can we make?

Total # of objects $n = 4$

we have a repetition, $a = 3$

$\Rightarrow$ total permutations $= \frac{n!}{a!}$

$= \frac{4!}{3!} = 4$

Example

How many permutations can we make of the words:

i) Vancouver 2 v's

$\frac{n!}{a!} = \frac{9!}{2!} = 181{,}440$

ii) Access

$n = 6$

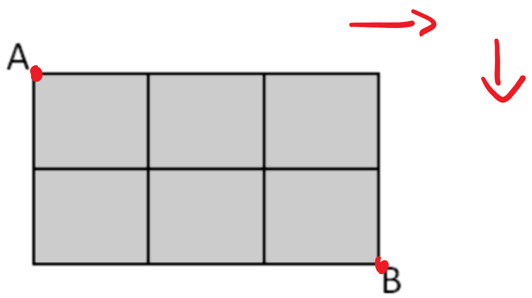
$b = 2$

$a = 2$

$\Rightarrow \frac{n!}{a!b!} = \frac{6!}{2!2!} = 180$

Permutations have an application that can help us determine the number of routes to a location (think Google maps).

Consider the following map where we want to get from point A to point B and every move must bring us closer to B:



If every move has to bring us closer to B, we will always need to move 3 times to the east and 2 times to the south.

n=5

You can think of this as "EEESS" and any permutation will bring us from A to B.

a=3 b=2 ☆ repetitions!! ☆

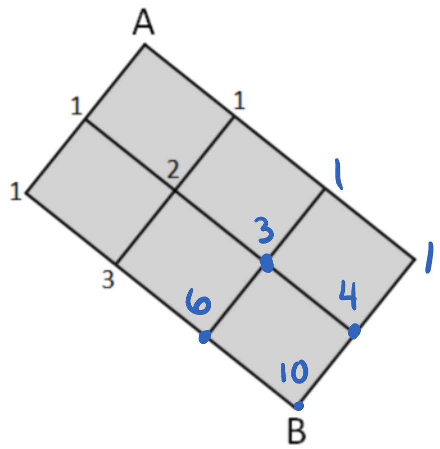
How many possible routes are there?

$n=5$
$a=3$
$b=2$

$$\left.\begin{matrix} n=5 \\ a=3 \\ b=2 \end{matrix}\right\} \text{total routes} = \frac{n!}{a!\,b!} = \frac{5!}{3!\,2!} = 10 \text{ possible routes.}$$

Let's consider an alternative strategy:

We'll rotate the map and number the amount of routes we can take to each point:



What's the pattern for determining the number of routes to any given point?

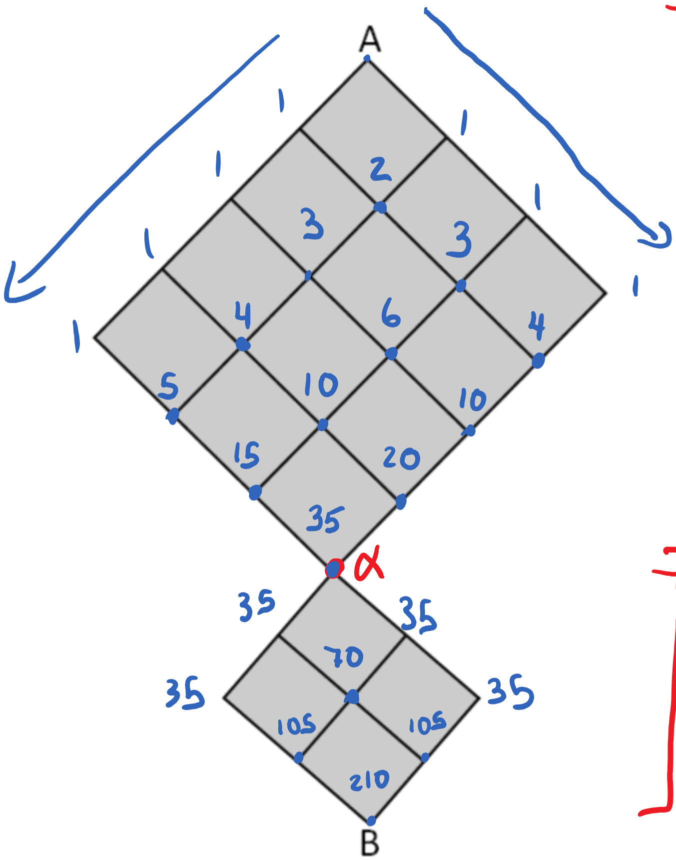
Add the numbers attached above the point.

Fill in the rest of the map. Do we get the same answer?

Yep.

Example

Determine the number of different routes to travel from point A to point B.



Alternatively

$A \to \alpha$

3 East
4 South

$\Rightarrow \dfrac{n!}{a!b!}$

$n=7$
$a=3$
$b=4$

$= \dfrac{7!}{3!4!} = 35$

$\alpha \to B$

2 East
2 South

$\Rightarrow \dfrac{n!}{a!b!}$

$n=4$
$a=2$
$b=2$

$= \dfrac{4!}{2!2!} = 6$

Total $= 35 \times 6 = \boxed{210 \text{ routes}}$