2.6 - Combinations Pt. II

August 29, 2019 1:42 PM

This section will deal with combination problems that deal with key words such as "at least," and "at most," as well as working backwards.

Imagine there is a committee of 11 students (5 males and 6 females) who want to form another smaller committee of five people. How many ways can the committee be arranged if there is at least one female?

Here, order doesn't matter, so it's a combination. One way to set this solution up is to add together the following possibilities:

Ways for 1F & 4M	Ways for 2F & 3M	Ways for 3F & 2M	Ways for 4F & 1M	Ways for 5F & 0M
${}_6C_1 \times {}_5C_4$	${}_6C_2 \times {}_5C_3$	${}_6C_3 \times {}_5C_2$	${}_6C_4 \times {}_5C_1$	${}_6C_5 \times {}_5C_0$
$6 \times 5 = 30$	150	200	75	6

$30 + 150 + 200 + 75 + 6 = 461$ ways.

Certainly, this takes quite some time to calculate. The question would be much easier to solve if there were no restrictions on the sex of the committee members (${}_{11}C_5$).

In some questions, however, it may be easier to calculate a solution by solving the problem as if there were no restrictions, and subtracting what's known as the complement.

The "complement" in a combination problem are the arrangements from the left over non-restricted answer after you use the restricted answer.

Using the last question as an example,

$$\underbrace{\begin{array}{c}({}_{11}C_5)\\ \text{\# of ways to select} \\ \text{all sub-committees}\end{array}}_{\text{unrestricted}} = \underbrace{\begin{array}{c}\text{\# of ways to select} \\ \text{At least 1 F}\end{array}}_{\text{restricted answer}} + \underbrace{\begin{array}{c}\text{\# of ways to select} \\ \text{0 F \& 5 M}\end{array}}_{\text{complement}}$$

If we use algebra to isolate the answer we actually wanted from the last question, the question becomes a lot easier to calculate:

# of ways to select all sub-committees	=	# of ways to select At least 1 F	+	~~# of ways to select 0 F & 5 M~~
− # of ways to select 0F + 5m				~~− # of ways to select 0F + 5m~~

# of ways to select at least 1 F = # of ways to select ALL committees − # of ways to select 0F + 5m

Example

Using a regular deck of cards, how many five card hands can be made containing at least one club?

restricted answer = unrestricted answer − complement

- In a deck, there are 13 of each suit (S, C, D, H).
- 52 cards in a deck.
- Taking away all clubs: $52 - 13 = 39$ cards.

[Hands w/ at least 1 club] = [All 5 card hands] − [Hands w/ 0 clubs]

$$= \left[{}_{52}C_5\right] - \left[{}_{39}C_5\right]$$

$$= 2{,}598{,}960 - 575{,}757$$

= 2,023,203 possible hands

Some combinations are equivalent. For example, let's say two people have an identical collection of ten toy cars.

The first person wants to display two cars on their table, and the other wants to display eight. Assuming both people don't care about the order they're displayed, how many combinations are there for each person?

1^{st}: ${}_{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10!}{8!2!} = 45$

2^{nd}: ${}_{10}C_8 = \frac{10!}{(10-8)!8!} = \frac{10!}{2!8!} = \frac{10!}{8!2!} = 45$

You can consider the "on the table" vs. "off the table" groups. The "off the table" group for 1 person is equivalent to the "on the table" group of the other, and vice-versa.

When there is a situation like this, the general rule is:

$$\boxed{{}_nC_r = {}_nC_{n-r}}$$

Finally, sometimes you will be asked to find "n" in a question after given the total number of possibilities.

For example, if two hockey teams shook hands after a game and there were 300 total unique handshakes, how many total players were there?

Recall the formula for combinations:

here,

${}_nC_r$ = total handshakes = 300

n = # of players = n

r = # of players needed for a handshake = 2

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

$$\times 2! \quad (300) = \frac{n!}{(n-2)!\cancel{(2)!}} \quad \cancel{\times 2!}$$

$$600 = \frac{n!}{(n-2)!}$$

expand until (n-2)!

$$600 = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$$

$$600 = n(n-1)$$

Guess + Check:

① and ② are 2 numbers that are 1 away from each other (consecutive).

↳ 24×25 = 600.

"n" is larger than "n-1", ∴ The larger number is "n".

∴ There were 25 players